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## Extensive form of equilibrium nonextensive statistics

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It is argued that, in nonextensive statistical mechanics with Tsallis entropy, the factorization of compound probability over subsystems is a consequence of the existence of thermodynamic equilibrium in the composite system and should be respected by all exact calculations concerning equilibrium subsystem. Using nonadditive energy satisfying this factorization, we propose an additive formalism of nonextensive statistical mechanics with additive  $q$ -deformed physical quantities and exponential distributions. This formalism leads to exact quantum gas distributions different from those given by factorization approximation with additive energy. The fermion distribution of present work shows similar characteristics to the distribution of strongly correlated electrons given by numerical analysis with Kondo t-J model.

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### I. INTRODUCTION

In this paper we will discuss some problems of nonextensive statistical mechanics (NSM) relevant to the factorization of compound probability into product of single body probability :

$$\rho = \prod_{n=1}^N \rho_n, \quad (1)$$

where  $N$  is the number of subsystems in the system of interest,  $\rho$  is the  $q$ -exponential distribution (QED) :  $\rho \propto [1 - (1 - q)\beta H]^{\frac{1}{1-q}}$  ( $[\cdot] \geq 0$ ), as given by the maximization of Tsallis entropy  $S_q = -\text{Tr} \frac{\rho - \rho^q}{1-q}$  (Boltzmann constant  $k_B = 1$  and  $q > 0$ ) under some constraints

[1–3]. This factorization Eq.(1) has been viewed as a result of the independence of noninteracting subsystems having additive energy, just as in Boltzmann-Gibbs statistics (BGS) supposing short range interactions, and caused confusions in some theoretical studies of NSM and its applications to many-body problems. On the basis of a new idea relating Eq.(1) to thermodynamic equilibrium, we will argue that these confusions can be avoided if we introduce suitable nonadditive thermodynamic variables satisfying Eq.(1). Some theoretical consequences for quantum distributions of this “equilibrium version” of NSM will be studied.

Due to the necessity of defining additive average value of some extensive  $q$ -deformed thermodynamic variables, the discussions will be made within the formalism of incomplete statistics (IS) with  $\text{Tr}\rho^q = 1$  and normalized average  $\bar{x} = \text{Tr}\rho^qx$  [3]. The reader will find that the quantum distributions of IS indeed show some particular properties already noticed with strongly correlated electrons.

## II. ABOUT INCOMPLETE NORMALIZATION

IS as a alternative version of NSM was originally motivated by some theoretical peculiarities in the last Tsallis version of NSM based on the conventional normalization and unnormalized expectations [3].

The basic assumption of IS is that our knowledge about physical systems is in general incomplete due to unknown space-time correlations or the effects of known interactions which can not be exactly studied (such as chaos). In this case, probability distributions are incomplete, i.e.,  $\text{Tr}\rho = Q \neq 1$  [4] (or  $\sum_{i=1}^w p_i = Q$  where  $w$  is only the number of accessible states in phase space). One can only write  $\text{Tr}F(\rho) = 1$  where  $F$  is certain function of  $\rho$ . In the case of complete or approximately complete distribution (such as in BGS),  $F$  is identity function. Recently, in order to overcome some of theoretical difficulties of NSM in keeping the framework prescribed by Tsallis entropy, we proposed [3]  $F(\rho) = \rho^q$  so that

$$\text{Tr}\rho^q = 1, \quad (2)$$

where  $q$  is the *incompleteness index* [3]. Since  $\rho < 1$ , we have to set  $q \in [0, \infty]$ .  $q = 0$  should be avoided because it leads to  $\rho = 0$  for all states. We note that Eq.(2) has been successfully employed to deduce some power laws based on Rényi's entropy [5].

This kind of *incomplete normalization* is possible whenever the phase space is partially known or accessible. With a fractal or chaotic phase space, e.g., a complete calculation of probability becomes in general impossible. In this sense, a plausible justification of Eq.(2) may be inspired by a work of Tsallis [6] discussing nonadditive energy and probability distributions on fractal supports, although at that stage the work was not connected to anomalous normalization like Eq.(2). In that work, considering some simple self-similar fractal structures (e.g. Cantor set), one can obtain :

$$\sum_{i=1}^W \left[ \frac{V_i(k)}{V(0)} \right]^{d_f/d} = 1 \quad (3)$$

where  $V_i(k)$  may be seen as the segments of the fractal structure at a given iteration of order  $k$ ,  $V(0)$  a characteristic volume of the fractal structure embedded in a  $d$ -dimension Euclidean space,  $d_f = \frac{\ln n}{\ln m}$  is the fractal dimension,  $n$  the number of segments replacing a segment of the precedent iteration,  $m$  the scale factor of the iterations and  $W = n^k$  the total number of segments at the  $k^{th}$  iteration. If we suppose that the fractal structure with  $k \rightarrow \infty$  is a phase space containing homogeneously distributed points, the *exact microcanonical probability distribution* of the  $k^{th}$  iteration can be defined as

$$p_i = \frac{V_i(k)}{V} = \frac{V_i(k)}{\sum_i^W V_i(k)}$$

where  $V$  is the total volume of the phase space. This distribution obviously sums to one. The problem is that  $V$  is an indefinite volume as  $k \rightarrow \infty$  and impertinent for exact probability definition. In addition,  $V$  is not differentiable and contains inaccessible points. Thus exact summation in  $V$  would be impossible. Now if we define  $p_i = \frac{V_i(k)}{V_0}$  as a physical or effective distribution, then we have

$$\sum_{i=1}^W p_i^{d_f/d} [V_0/V(0)]^{d_f/d} = 1,$$

where  $V_0$  is a *completely accessible and infinitely differentiable support* on which the calculation of  $p_i$  is possible. If we choose  $V_0 = V(0)$ , we can write Eq.(2) with  $q = d_f/d$ . The conventional normalization  $\sum_{i=1}^W p_i = 1$  can be recovered when  $d_f = d$ .

The above example is only a case of equiprobable distribution on simple fractal structure, but it illustrates very well the possibility that, in complex cases, a physical probability may not sum to one and may sum to unity only through a kind of power normalization, which, pertinent and useful for incomplete distributions, is consistent with the discussions of reference [6] on the mass calculation and the information consideration in porous structures.

### III. FACTORIZATION OF COMPOUND PROBABILITY AND THERMODYNAMIC EQUILIBRIUM

In NSM, there are two major problems connected tightly to the factorization of compound probability. The first concerns the application of NSM to many-body systems via one-body distribution. NSM is originally intended to describe complex systems with long range interactions or fractal structure of space-time showing nonextensive phenomena. So from the beginning of this theory, Eq.(1) is supposed for composite systems containing  $N$  *statistically independent subsystems* in order to elicit the nonextensive character by following relation :

$$\ln[1 + (1 - q)S_q] = \sum_{n=1}^N \ln[1 + (1 - q)S_q(n)] \quad (4)$$

For  $N = 2$ ,  $S_q = S_q(1) + S_q(2) + (1 - q)S_q(1)S_q(2)$  as one often finds in the literature. Due to this *independence*, it has been believed by many that exact calculations within NSM should use the additive hamiltonian  $H_0 = \sum_{n=1}^N H_n$ , where  $H_n$  is the hamiltonian of  $n^{th}$  subsystem [2,7–11]. However, this hamiltonian is not compatible with neither Eq.(1) nor Eq.(4) since these equations applied to QED mean [1,12,13]:

$$\begin{aligned} H &= \sum_{n=1}^N H_n + \sum_{k=2}^N [(q - 1)\beta]^{k-1} \sum_{n_1 < n_2 < \dots < n_k} \prod_{j=1}^k H_{n_j} \\ &= H_0 + H_c, \end{aligned} \quad (5)$$

where  $\beta$  is the inverse temperature. In order to reconcile  $H_0$  and Eq.(1), a so called *factorization approximation* is proposed [14] by neglecting the second term at the right hand side of Eq.(5). This approximation has been, explicitly or not, employed in most of the applications of NSM [13–18] via one-body QED. These applications certainly shows the usefulness of one-body QED, but the approximation neglecting the correlation energy by supposing sometimes weak interacting dilute particles [19] is not a reassuring basis. Indeed, some recent works show that the correlation energy ( $H_c$ ) given by the second term of Eq.(5) is in general not negligible [12] and that the partition function given by using additive energy is completely different from that given by using Eq.(5) when  $N$  is large [11]. So a doubt arises about the connection between independence of subsystems and additive energy. Recently, an interesting idea is forwarded to define a “quasi-independence” according to nonadditive energy Eq.(5) in order to apply NSM to turbulence flow problems [19]. As a matter of fact, this proposal implies rejection of classical independence for Eq.(1).

The second problem connected to probability factorization is the establishment of zeroth law and the definition of temperature for NSM. It was believed that the zeroth law of thermodynamics was absent within NSM [20] due to the paradox between Eq.(1) and the additive energy. Recently, a series of works have been published on this issue [21] claiming the establishment of zeroth law and the definition of a generalized temperature on the basis of additive hamiltonian  $H_0$  and Eq.(4) by neglecting  $H_c$ . It is evident that the paradox mentioned above persists behind this approximate zeroth law.

The central question is : Eq.(1) certainly implies independence of noninteracting systems for *BGS*, but does it mean the same thing for NSM? Very recently, Abe [22] proposed a general pseudoadditivity for entropy required by the existence of thermal equilibrium in composite nonextensive systems. For a system containing  $N$  subsystems, the pseudoadditivity is :

$$\ln[1 + \lambda_S f(S)] = \sum_{n=1}^N \ln[1 + \lambda_S f(S_n)], \quad (6)$$

where  $f$  is certain differentiable function satisfying  $f(0) = 0$  and  $\lambda_S$  a constant depending

on the nature of the system of interest. On the other hand, Eq.(6) applied to Tsallis entropy means  $f(S) = S$  and  $\lambda_S = 1 - q$  [22], which directly leads to  $\ln \text{Tr} \rho^q = \sum_{i=1}^N \ln \text{Tr} \rho_i^q$  or Eq.(1) (i.e. with classical probability  $p_i$  of the state  $i$ ,  $(p_i p_j)^q = p_{ij}^q$  means  $p_i p_j = p_{ij}$ ). So Eq.(1) *has nothing to do with statistical independence of subsystems*. It is *a consequence of the existence of thermodynamic equilibrium* and must be rigorously respected by all exact calculations. Equilibrium energy has been proved [23] to satisfy the same kind of pseudoadditivity as Eq.(6) ( $S$  is replaced by  $H$ ). If we choose  $f(H) = H$  and  $\lambda_H = (q - 1)\beta$ , we get

$$\ln[1 + (q - 1)\beta H] = \sum_{n=1}^N \ln[1 + (q - 1)\beta H_n] \quad (7)$$

which is just Eq.(5) satisfying Eq.(1). In this way, the zeroth law becomes evident and a temperature can be straightforwardly defined at maximum entropy and minimum energy [3,12].

## IV. ADDITIVE FORMALISM OF NSM

### A. Information measure

The  $g$ -logarithmic information measure

$$I_\nu = \frac{(1/\rho)^\nu - 1}{\nu} \quad (8)$$

is a nonadditive generalization of Hartley formula  $I = \ln(1/\rho)$  and can be employed to deduce Tsallis entropy [1,3,24].  $I_g$  or  $I$  is the information needed to specify at which state the system is localized.  $\nu$  equals  $1 - q$  or  $q - 1$ , depending on the normalization procedures of  $\rho$  [25]. Using Eq.(1), we get :

$$\ln(1 + \nu I_\nu) = \sum_{n=1}^N \ln(1 + \nu I_\nu^{(n)}) \quad (9)$$

where  $I_\nu^{(n)}$  is the information needed to specify the  $n^{th}$  subsystem. This pseudoadditivity is evident if we recast the generalized Hartley formula Eq.(8) as follows

$$I_\nu = \frac{e^{-\nu \ln \rho} - 1}{\nu} = \frac{e^{\nu I} - 1}{\nu} \quad (10)$$

where

$$I = \ln \frac{1}{\rho} = \frac{\ln(1 + \nu I_\nu)}{\nu}. \quad (11)$$

can be referred to as  $q$ -deformed information measure and is additive supposed Eq.(1). It is noteworthy that this  $I$  is not the quantity of Hartley information if  $\rho$  is a nonextensive distribution for  $\nu \neq 0$ .

## B. Canonical ensemble

Now let us define an additive entropy  $S$  as follows :

$$S = \text{Tr} \rho^q \ln \frac{1}{\rho} \quad (12)$$

and an additive  $q$ -deformed “hamiltonian”

$$h = \frac{\ln[1 + (q - 1)\beta H]}{(q - 1)\beta}. \quad (13)$$

So Eq.(7) becomes

$$h = \sum_{n=1}^N h_n. \quad (14)$$

This means following transformations :

$$H = \frac{e^{(q-1)\beta h} - 1}{(q - 1)\beta}, \quad H_n = \frac{e^{(q-1)\beta h_n} - 1}{(q - 1)\beta} \quad (15)$$

and

$$\rho = \frac{1}{Z} [1 + (q - 1)\beta H]^{1/(1-q)} = \frac{1}{Z} e^{-\beta h}, \quad (16)$$

where  $Z^q = \text{Tr} e^{-q\beta h}$  [3]. It should be noticed that, when addressing a system of  $N$  particles, we have to write  $H_n = \frac{p_n^2}{2m} + V_n$  for single particle so that  $h_n = \frac{\ln[1 + (q - 1)\beta(\frac{p_n^2}{2m} + V_n)]}{(q - 1)\beta}$  where  $\frac{p_n^2}{2m}$  is the classical kinetic energy and  $V_n$  is the potential energy. It is clear that  $H_n$ , instead of  $h_n$ , is the physical energy. When  $q = 1$  ( $H_c = 0$ ), we recover  $H_n = h_n$  and  $H = h = \sum_{n=1}^N \frac{p_n^2}{2m}$ . The  $q$ -deformed internal energy  $u$  is defined as follows :

$$u = \text{Tr} \rho^q h. \quad (17)$$

We can easily show that the distribution Eq.(16) can be yielded by the maximum of the additive “entropy”  $S$  (which surely exists due to the monotonic relation between  $I$  and  $I_\nu$ ) under the constraints of Eq.(17) and incomplete normalization  $\text{Tr} \rho^q = 1$ . It is easy to verify that  $S = \ln Z + \beta u$  and, via the zeroth law,  $\frac{\partial S}{\partial u} = \beta = 1/T$ . The “first law” is given by  $du = TdS - pdV$  where  $p$  is  $q$ -deformed pressure and  $V$  the volume of the system which is chosen to be additive here. The  $q$ -deformed Helmholtz free energy  $f$  is defined as  $f = u - TS = -T \ln Z$  and can be connected to the nonadditive one  $F_q = -T \frac{Z^{1-q}-1}{1-q}$  [1,3] as follows :

$$f = \frac{\ln[1 + (q-1)\beta F_q]}{(q-1)\beta}. \quad (18)$$

So  $p = -(\frac{\partial f}{\partial V})_T = P/Z^{1-q}$  where  $P = -(\frac{\partial F_q}{\partial V})_T$  is the real pressure. In this scenario, the thermodynamic equilibrium of a system  $C$  containing two equilibrium systems  $A$  and  $B$  satisfying  $V(C) = V(A) + V(B)$  corresponds to  $\beta(A) = \beta(B)$  and  $p(A) = p(B)$ . This implies that  $P(A) \neq P(B)$  if  $Z(A) \neq Z(B)$ . This is because we have supposed nonadditive energy and additive volume. As a matter of fact, in this formalism, if we want that  $P(A) = P(B)$  at equilibrium, we must accept nonadditive volume and additive  $q$ -deformed volume  $v$  with which the first law becomes  $du = TdS - Pdv$ . This means :  $P = -(\frac{\partial F_q}{\partial V})_T = -(\frac{\partial f}{\partial v})_T$ . This relation can help to deduce  $v - V$  relation. We also have  $v = (\frac{\partial g}{\partial P})_T$  where the  $q$ -deformed Gibbs energy is given by  $g = f + Pv$ . Since  $v$  is additive,  $V$  will be nonadditive if  $v$  is not a linear function of  $V$ . We will come back to this issue later in this paper.

### C. Grand canonical ensemble

It is known that the grand canonical ensemble  $QED$  has been given by [1,9,14] :

$$\rho \propto [1 - (1-q)\beta(H - \mu N)]^{\frac{1}{1-q}} \quad (19)$$

for  $N$  identical particle systems, where  $\mu$  is chemical potential. This distribution has been widely used for quantum particle systems [14–16,26]. But the zeroth law has never been

rigorously established for this ensemble. In a previous work [26], one of the authors of present paper deduced exact quantum distributions on the basis of Eq.(19) and following relation suggested by Eq.(1) :

$$\rho \propto [1 - (1 - q)\beta(H_n - \mu)]^{\frac{N}{1-q}}. \quad (20)$$

In the framework of IS [3,26], the exact distributions are given by

$$\bar{n}_k = \frac{1}{[1 + (q - 1)\beta(e_k - \mu)]^{\frac{q}{q-1}} \pm 1} \quad (21)$$

where  $e_k$  is the energy of one-particle state  $k$  and “+” and “-” correspond to fermions and bosons, respectively. Now we show that this distribution can be written in exponential form just as for conventional noninteracting quantum gases and that the zeroth law can be rigorously verified.

Let us suppose  $\beta = \frac{\beta'}{1+(q-1)\beta'\mu} = \beta'[1 - (q - 1)\beta'\mu']$  (or  $\beta' = \frac{\beta}{1-(q-1)\beta\mu}$ ) and  $\mu = \frac{\mu'}{1+(1-q)\beta'\mu'}$  (or  $\mu' = \frac{\mu}{1+(q-1)\beta'\mu} = \mu[1 - (q - 1)\beta\mu]$ ) which imply  $\beta'\mu' = \beta\mu$ . Eq.(20) can be recast as :

$$\begin{aligned} \rho &= \frac{1}{Z} [1 - (1 - q)\beta'H_n]^{\frac{N}{1-q}} [1 + (1 - q)\beta'\mu']^{\frac{N}{1-q}} \\ &= \frac{1}{Z} e^{-N\beta(h_n - \omega)} \end{aligned} \quad (22)$$

where  $\omega = \frac{\ln[1+(1-q)\beta'\mu']}{(1-q)\beta'}$  and  $h_n = \frac{\ln[1+(q-1)\beta'H_n]}{(q-1)\beta'}$ .  $Z^q = \text{Tr}[1 - (1 - q)\beta(H - \mu N)]^{\frac{q}{1-q}} = \{\text{Tr}[1 - (1 - q)\beta'H_n]^{\frac{q}{1-q}} [1 + (1 - q)\beta'\mu']^{\frac{q}{1-q}}\}^N = (\text{Tr}e^{-q\beta(h_n - \omega)})^N = z^N$  and where  $z$  is one-particle partition function. Just as for canonical ensemble, this exponential distribution can be shown to be the result of the maximization of  $S$  under the constraint  $\bar{N} = \text{Tr}\rho^q N$  in addition to Eq.(17) and incomplete normalization. Now Eq.(21) can be written as

$$\bar{n}_k = \text{Tr}\rho^q n_k = \frac{1}{e^{q\beta'(\epsilon_k - \omega)} \pm 1} = \frac{1}{[1 + (q - 1)\beta'e_k]^{\frac{q}{q-1}} [1 - (q - 1)\beta'\mu']^{\frac{q}{q-1}} \pm 1} \quad (23)$$

where  $\epsilon_k$  is the eigenvalue of  $h_n$ . From Eq.(21), we see that, for free particles (*in the sense that we do not write the energy of interaction between particles in the hamiltonian and let it be “absorbed” in the nonextensive part of energy  $H_c$  and related to  $q$  different from unity*), we have to set  $q < 1$  to ensure positive  $[1 + (q - 1)\beta(H_n - \mu)]$  for fermions when  $T \rightarrow 0$ .

This means that, at low temperatures, there will be few fermions beyond Fermi-energy. For bosons with  $\mu < 0$ , we have to put  $q > 1$ .

It is straightforward to write  $S = \ln Z + \beta' u + \beta' \omega \bar{N}$  and, with the method employed in references [3,12], to show that, for a system  $C$  containing two equilibrium systems  $A$  and  $B$  satisfying  $\bar{N}(C) = \bar{N}(A) + \bar{N}(B)$ ,  $\beta'(A) = \beta'(B)$  and  $\omega(A) = \omega(B)$  thus  $\mu'(A) = \mu'(B)$ ,  $\mu(A) = \mu(B)$  and  $\beta(A) = \beta(B)$  or  $T(A) = T(B)$ . The zeroth law is verified. One may ask why we identify  $\beta$ , instead of  $\beta'$ , to real temperature. The possible reasons are the following : 1)  $\beta$  is the Lagrange multiplier of the constraint on real energy in entropy maximization; 2) if  $\beta'$  is inverse temperature, then  $\mu'$  must be chemical potential, which makes it impossible to get distribution Eq.(22) by real entropy maximization with the constraint on  $\bar{N}$ ; 3)  $e_f$  would be different from the chemical potential  $\mu'$  and equal to  $\frac{\mu'}{1+(1-q)\beta'\mu'}$  which inevitably drops to zero when  $T \rightarrow 0$ .

Now we will focus the discussion on 2D fermion distribution. According to Eq.(21) the Fermi energy  $e_{f0}$  at  $T = 0$  is given by  $e_{f0} = \frac{2\pi\hbar^2\sigma}{m}$  where  $\sigma$  is the particle density and  $\hbar$  Planck constant.

When  $T > 0$ , the summation  $\bar{N} = \sum_k \bar{n}_k$  can not be calculated for arbitrary  $q$  to give explicit expression of  $e_f$ . So we have recourse to numerical calculation of  $q$ -dependence of  $e_f$  for given temperatures (Figure 1) and  $T$ -dependence for given  $q$  values (Figure 2). We see that, for the approximate distributions functions (ADF)  $n_k = 1/\{[1+(q-1)\beta(e_k-\mu)]^{\frac{1}{q-1}}+1\}$  deduced from Eq.(19) with factorization approximation and additive energy [14],  $e_f$  depends only slightly on  $q$ . On the other hand, the  $e_f$  of IS in the present work shows a strong increase with decreasing  $q$  up to two times  $e_{f0}$  of conventional Fermi-Dirac distribution (CFD). This  $e_f$  increase has been indeed noticed through numerical calculations for strongly correlated heavy electrons on the basis of tight-binding Kondo lattice model [27,28]. In Figure 2, we show that  $e_f$  of IS does not monotonically decrease with increasing temperature, in contrast with the  $e_f$  behavior of CFD. This kind of non monotonic temperature evolution of  $e_f$  was also seen through numerical work for correlated electrons in two-dimensional  $t - J$  model [29]

The IS distribution given by Eq.(21) is plotted in Figure 3 for  $T = 100K$ . The particle density  $\sigma$  is chosen to given  $e_{f_0} = 1$  eV. As expected, the distribution changes drastically with  $q$ . When  $q$  decreases, we notice a flattening of the distribution with always a sharp drop of occupation number  $n$  at  $e_f$  which increases. This flattening is also noticed in numerical calculations for strong coupling electrons [27,28].

The grand canonical partition function  $Z$  can be calculated to give the  $q$ -deformed grand potential  $\bar{\omega} = -T' \ln Z = \frac{T'}{q} \sum_k \ln(1 - \bar{n}_k)$  as usual, where  $T' = 1/\beta' = T + (1 - q)\mu$ . The grand potential is given by :

$$\Omega = \frac{e^{(q-1)\beta'\bar{\omega}} - 1}{(q-1)\beta} = T \frac{\prod_k (1 - \bar{n}_k)^{\frac{q-1}{q}} - 1}{(q-1)}. \quad (24)$$

In this  $q$ -deformed extensive formalism, Euler theorem applies just as in BGS. So we have  $g = \omega \bar{N} = u + pV - T'S$ . Compare this with  $S = \ln Z + \beta'u - \beta'\omega \bar{N}$  to obtain the following equations of state :

$$pV = T' \ln Z = \frac{T'}{q} \sum_k \ln(1 - \bar{n}_k). \quad (25)$$

## V. DISCUSSION AND CONCLUSION

The formalism of NSM presented here is required by the existence of thermodynamic equilibrium or by Eq.(1) for nonextensive systems described by Tsallis entropy. Theoretically the formalism is self-consistent. Experimental or numerical evidences are needed to verify the thermodynamic relations. In this framework, all the successful applications of NSM conforming with Eq.(1) are still valid. But the approximate applications carried out for many-body systems using additive energy as exact hamiltonian (not consistent with the existence of thermodynamic equilibrium) should be carefully reviewed.

As mentioned above, we have noticed similar properties between the IS fermion distribution and that of strongly correlated electrons [27,28]. This similarity shows the merit of NSM in describing strong interacting systems. On the other hand, we noticed that a flattening of  $n$  drop at  $e_f$  with increasing correlation, observed experimentally and numerically

with weakly correlated electrons [27,28,30–32], is absent within NSM fermion distributions which always show very sharp  $n$  drop at  $e_f$  at low temperatures. A detailed study of this problem will be presented in another paper of ours.

It is worth mentioning that, in the present work, energy is nonadditive to satisfy the requirement of thermodynamic equilibrium. Nonadditive energy can happen if interaction is no longer of short-range and not localized only between the containing walls of subsystems. But in the literature, there are rarely explicit expressions of nonadditive energy. One of explicit examples is the long-range Ising model [33] where the internal energy  $U(N, T)$  may proportional to  $N^c$  ( $c$  is a constant), instead of  $N$ , the number of spins in the system. This energy can be shown to satisfy Abe’s pseudoadditivity for energy [23] if we choose  $f(U)$  to be proportional, e.g., to  $N$  (with  $\lambda_H = 0$ ) or to  $e^N - 1$  (with  $\lambda_H = 1$ ).

Indeed, theoretically, nonadditive physical quantities (energy, volume ...) is not evident within the statistics with complete distributions because all possible states (all points in phase space) are counted and summed here. But from the viewpoint of incomplete statistics, nonadditivity may be interpreted as a consequence of incomplete summation of state points in phase space due to the incompleteness of our knowledge about the physical systems [3].

Nevertheless, the fact that the correlation energy  $H_c$  of NSM depends on temperature, as shown in Eq.(5) or Eq.(7), is not an easy aspect to be understood. A possible interpretation is that these nonadditive equations are required or prescribed by thermal equilibrium with Tsallis entropy and so naturally change with temperature. This implies that the effect of correlations may depend on temperature.

Summing up, within the framework of *incomplete statistics*, it is argued that the nonextensive thermostatistics should be based on the factorization of compound probability suggested, not by “independence” of noninteracting systems, but by *existence of thermodynamic equilibrium in interacting systems having Tsallis entropy*. So this factorization must be viewed as a fundamental hypothesis of NSM and rigorously satisfied by all exact calculations relative to equilibrium systems. On this basis, we have elaborated an additive formalism of NSM based on the maximization of an additive deformed entropy subject to

constraints on additive particle number and additive  $q$ -deformed energy. The IS quantum distributions of this formalism are compared with the distributions previously obtained by using complete probability and additive energy in factorization approximation and also with numerical results for strongly correlated electrons. It is shown that some effects of strong correlations : the flattening of the fermion distribution and the sharp cutoff of occupation number at  $e_f$  which shows strong increase with increasing interaction, can be observed in IS fermion distribution with decreasing  $q$  value.

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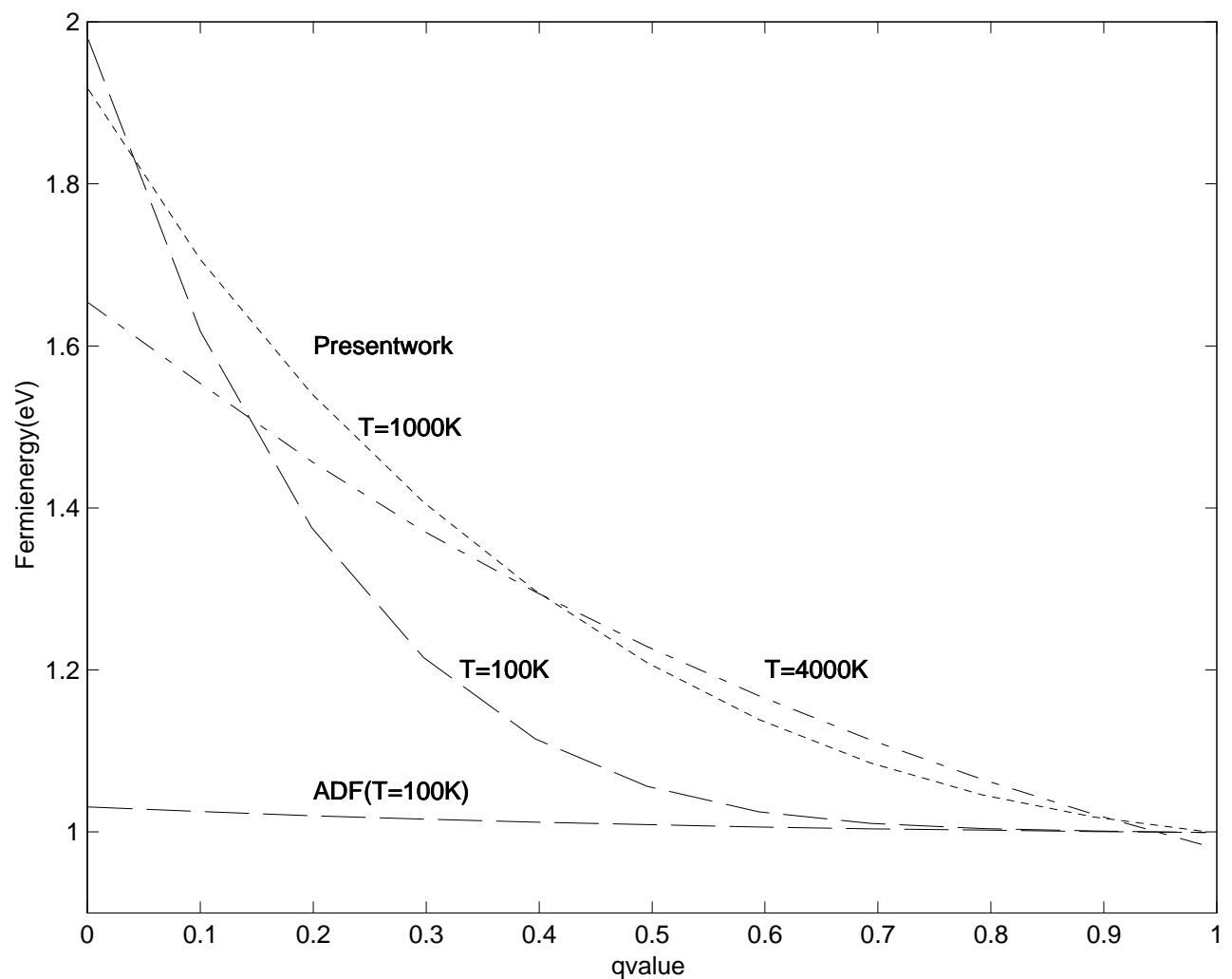
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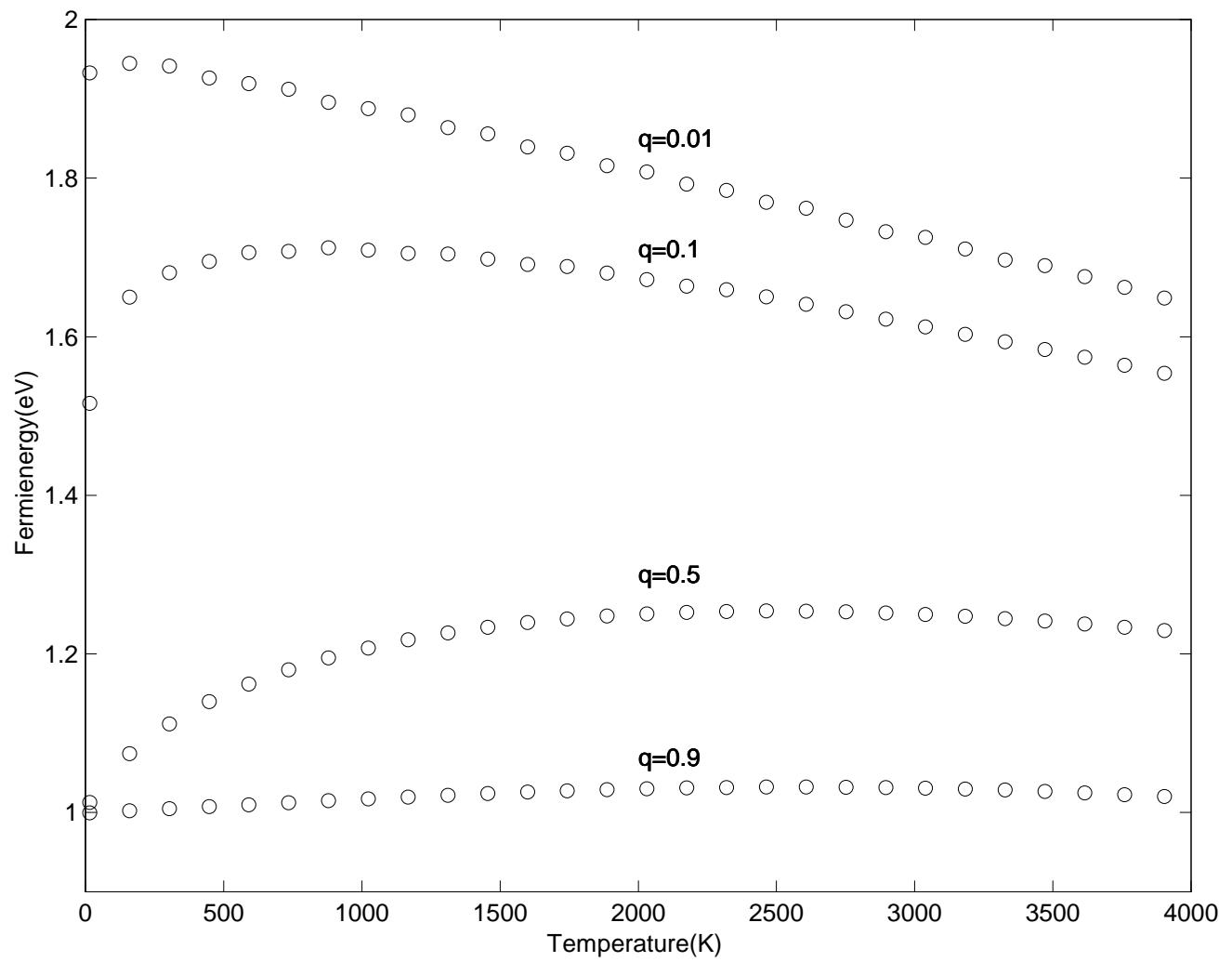
## Figures:

FIG. 1.  $q$ -dependence of Fermi energy  $e_f$  of IS quantum distribution in present work and of ADF given by the factorisation approximation with additive energy. The fermion density  $\sigma$  is chosen to give  $e_f^0 = 1$  eV for CFD distribution at  $T = 0$ . IS  $e_f$  shows strong increase with decreasing  $q$  up to two times  $e_{f0}$ . But ADF  $e_f$  depends only slightly on  $q$ . We also notice that the  $T$ -dependence of IS  $e_f$  is not monotonic as shown in Figure 2.

FIG. 2.  $T$ -dependence of Fermi energy  $e_f$  of IS quantum distribution in present work. The fermion density  $\sigma$  is chosen to give  $e_f^0 = 1$  eV for CFD distribution at  $T = 0$ . The  $T$ -dependence of  $e_f$  is in general not monotonic, in contrast with the classical decreasing behavior of  $e_f$  with increasing temperature. We notice that, at low temperature,  $e_f$  shows an increase with increasing  $T$ .

FIG. 3. Fermion distributions of ADF and of IS in present work. ADF distribution is only slightly different from that at  $q = 1$  (CFD one) even with  $q$  very different from unity. The IS distribution of present work changes drastically with decreasing  $q$ . As  $q \rightarrow 0$ , the occupation number tends to 1/2 for all states below  $e_f$  which increases up to 2 times  $e_{f0}$ .





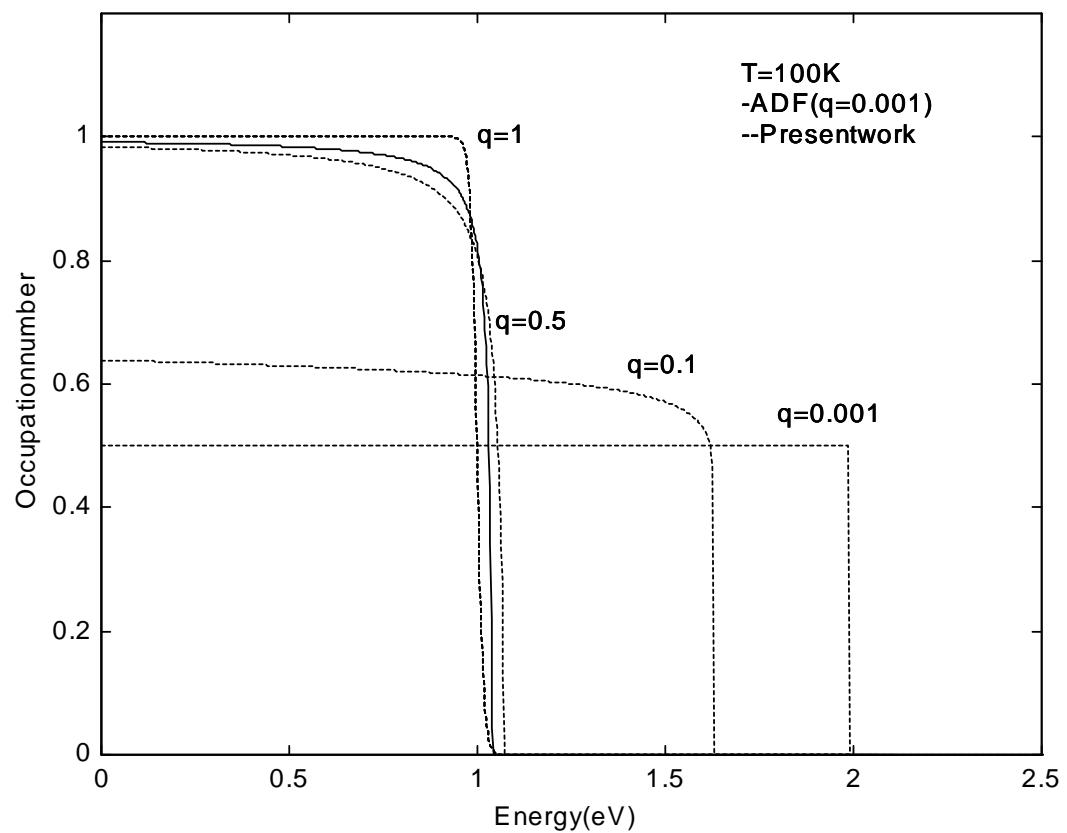


Figure 3 :